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A Class of Eight Order Iterative Methods for Solving Nonlinear Equations.

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Abstract

In this paper, we presented a new class of optimal eighth-order methods for solving simple roots of nonlinear equations. The class is developed by combining a special case of King's fourth-order method and Newton's method as a third step using the forward divided difference and multiplication of three weight function. Some numerical comparisons have been considered to show the performance of the proposed method.

Keywords: Convergence order; Efficiency index; Iterative methods; Nonlinear equations; Optimal method.

1. Introduction

In this paper, we develop iterative methods to find a simple root γ of the nonlinear equation $f(\gamma) = 0$, where $f : D \subset R \to R$ is a scalar function on an open interval D. It is well known that Newton's method is one of the best iterative method for solving a single nonlinear equation by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

Which converges quadratically in some neighbourhood of γ [9]. In the last years, many modified methods have been proposed to improve several iterative methods for solving nonlinear equations. Abbasbandy [1] and Chun [5] have proposed and studied several one-step and two-step iterative methods with higher order of convergence using the Adomian decomposition method [2]. Three-step iterative method proposed by J. Yun [15], which is a significant improvement of the method proposed by Noor and Noor [8].

A family of two steps is proposed by King [11], given by,

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= w_n - \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)}. \end{aligned} \tag{2}$$

Where $\beta \in R$ is a constant. In particular, the special method for $\beta = -0.5$ is as follows:

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{2f(x_n)-f(w_n)}{2f(x_n)-5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)}, \end{aligned} \tag{3}$$

the second step is taken from the second step of Bi et al. which is of fourth-order of convergence [4].

The efficiency index (EI) is defined by $E = p^{1/d}$, where p is the order of convergence and d is the number of total function and derivative evaluations per iteration [13]. According to the optimality, the optimal order of any multipoint iterative method is given by 2^{d-1} [6]. So, the efficiency index of Newton method, (NM), (1), is $2^{1/2} \approx 1.4142$ and for the optimal fourth order King method, (KM), (2), and method, (3), is $4^{1/3} \approx 1.5874$.

Theorem 1 [10]: Let $\delta_1(x), \delta_2(x), \ldots, \delta_m(x)$be iterative functions with the orders $q_1, q_2, \ldots, q_m$, respectively. Then the composition of iterative functions $\delta_1\left(\delta_2(\ldots(\delta_m(x))\ldots)\right)$, defines the iterative method of the order $q_1 q_2 \ldots q_m$.

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Using theorem 1, and adding the Newton's method as a third step to method (3), we have

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = w_n - \frac{2f(x_n)-f(w_n)}{2f(x_n)-5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \qquad (4)$$

The method (4) has $EI = 8^{1/5} \approx 1.5157$, and is not optimal. To reduce the number of functions evaluation of method (4) to four, by replacing $f'(z_n)$ to $\frac{f[x_n,z_n]f[w_n,z_n]}{f[x_n,w_n]}$ using the divided difference [12], to develop a family of optimal eight-order of convergence methods.

2. Development of Method and Convergence Analysis

The order of convergence of the proposed method (4) is eight but it is not optimal. To construct an optimal eighth-order method without using more evaluations, we present a new family of optimal eighth-order as follows, (HSM)

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = w_n - \frac{2f(x_n)-f(w_n)}{2f(x_n)-5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}\{H(s_1)\cdot K(s_2)\cdot B(s_3)\}, \qquad (5)$$

where $H(s_1), K(s_2), B(s_3)$, are three real-valued weight functions, and

$$s_1 = \frac{(fw)}{(fx)}, s_2 = \frac{(fz)}{(fw)}, s_3 = \frac{(fz)}{(fx)}. \qquad (6)$$

The weight functions $H(s_1), K(s_2)$ and $B(s_3)$ should be chosen such that the order of convergence of method (5) arrives at an optimal level of eight. In the following theorem we prove that method (5) has an optimal eighth-order of convergence under conditions for the weighted functions.

Theorem 2. Let γ in D be a simple root of a sufficiently differentiable function $f: D \subseteq R \to R$. If x_0 is sufficiently close to γ then the family of iterative methods (5) has an optimal eighth-order of convergence when

$H(0) = 1, H'(0) = 0, H''(0) = 0, H'''(0) = 6, |H^{(4)}(0)| < \infty,$
$K(0) = 1, K'(0) = 0, |K''(0)| < \infty,$
$B(0) = 1, B'(0) = 1.$

Proof: Let $e_n = x_n - \gamma$ be the error at the nth iteration, by Taylor expansion, we have

$$f(x_n) = f'(\gamma)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)]. \qquad (7)$$

Where $c_k = \frac{f^{(k)}(\gamma)}{k!f'(\gamma)}, k = 2,3, \dots$.

$$f'(x_n) = f'(\gamma)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + O(e_n^9)]. \qquad (8)$$

Dividing (7) by (8), gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + \dots + (19c_2c_7 - 7c_8 - 118c_5c_2c_3 + 348c_4c_3c_2^2 - 64c_2^7 - 64c_2c_4^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3 - 44c_6c_2^2 + 304c_3c_2^5 - 75c_4c_3^2 + 31c_5c_4 + 135c_2c_3^3 - 408c_3^2c_2^3)e_n^8 + O(e_n^9). \qquad (9)$$

Now, from (9), we have

$$w_n = \gamma + c_2{e_n}^2 + (-2c_2^2 + 2c_3){e_n}^3 + \dots + (64c_2^7 - 304c_2^5c_3 + 176c_2^4c_4 + 408c_2^3c_3^2 - 92c_2^3c_5 - 348c_2^2c_3c_4 - 135c_2c_3^3 + 44c_2^2c_6 + 118c_2c_3c_5 + 64c_2c_4^2 + 75c_3^2c_4 - 19c_2c_7 - 27c_3c_6 - 31c_4c_5 + 7c_8){e_n}^8 + O(e_n^9). \qquad (10)$$

From (10), we get

$$f(w_n) = f'(\gamma)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \dots + (144c_2^7 - 552c_2^5c_3 + 297c_2^4c_4 + 582c_2^3c_3^2 - 134c_2^3c_5 - 455c_2^2c_3c_4 - 147c_2c_3^3 + 54c_2^2c_6 + 134c_2c_3c_5 + 73c_2c_4^2 + 75c_3^2c_4 - 19c_2c_7 - 27c_3c_6 - 31c_4c_5 + 7c_8){e_n}^8] + O(e_n^9). \qquad (11)$$

Combining (7), (8), (9) and (11), we have

$$z_n = \gamma + (-c_2c_3){e_n}^4 + \dots + \left(-\frac{1515}{16}c_2^7 + 14c_3^2c_4 - 5c_2c_7 - 13c_3c_6 + \frac{573}{2}c_2^5c_3 - \frac{413}{4}c_2^4c_4 - 199c_2^3c_3^2 + 19c_2^3c_5 + 13c_2c_3^3 + 5c_2^2c_6 + 10c_2c_4^2 - 17c_4c_5 + 61c_2^2c_3c_4 + 20c_2c_3c_5\right){e_n}^8 + O(e_n^9). \qquad (12)$$

From (12), we get

$$f(z_n) = f'(\gamma)[(-c_2c_3){e_n}^4 + \dots + \left(-198c_2^3c_3^2 - \frac{1515}{16}c_2^7 + 14c_3^2c_4 - 5c_2c_7 - 13c_3c_6 + \frac{573}{2}c_2^5c_3 - \frac{413}{4}c_2^4c_4 + 19c_2^3c_5 + 13c_2c_3^3 + 5c_2^2c_6 + 10c_2c_4^2 - 17c_4c_5 + 61c_2^2c_3c_4 + 20c_2c_3c_5\right){e_n}^8] + O(e_n^9). \qquad (13)$$

From (7), (11) and (13), it can be easily to found

$$f[x_n, w_n] = f'(\gamma)[1 + c_2e_n + (c_2^2 + c_3){e_n}^2 + \dots + (-42c_2^2c_3^3 + 45c_2^3c_6 + 69c_2^2c_4^2 - 20c_2^2c_7 - 8c_2^3c_5 - 8c_3c_4^2 + 8c_2c_8 + 8c_3c_7 + 8c_4c_6 - 256c_2^6c_3 + 184c_2^5c_4 + 264c_2^4c_3^2 - 93c_2^4c_5 + 64c_2^8 - 6c_3^4 + 4c_5^2 + c_9 - 313c_2^3c_3c_4 + 116c_2^2c_3c_5 + 56c_2c_3^2c_4 - 36c_2c_3c_6 - 40c_2c_4c_5){e_n}^8] + O(e_n^9). \qquad (14)$$

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$$f[w_n,z_n] = f'(\gamma)[1 + c_2^2 {e_n}^2 + ... + \left(-22c_2c_3^3 - 20c_2^2c_6 - 18c_2c_4^2 + 6c_2c_7 + 30c_2^5c_3 - 64c_2^4c_4 - 22c_2^3c_3^2 + 40c_2^3c_5 - \frac{1}{8}c_2^7 + 80c_2^2c_3c_4 - 24c_2c_3c_5 + 12c_3^2c_4\right){e_n}^7] + O({e_n}^8). \quad (15)$$

$$f[x_n,z_n] = f'(\gamma)[1 + c_2e_n + c_3{e_n}^2 + ... + \left(30c_2^2c_3^3 + 5c_2^3c_6 + 13c_2^2c_4^2 - 5c_2^2c_7 - 12c_3^2c_5 - 13c_3c_4^2 + \frac{2547}{8}c_3 - 112c_2^5c_4 - \frac{533}{2}c_2^4c_3^2 + \frac{41}{2}c_2^4c_5 - \frac{1515}{16}c_2^8 + 4c_3^4 + c_9 + 83c_2^3c_3c_4 + 26c_2^2c_3c_5 + 36c_2c_3^2c_4 - 18c_2c_3c_6 - 22c_2c_4c_5\right){e_n}^8] + O(e_n^9). \quad (16)$$

By expanding $H(s_1), K(s_2), B(s_3)$ using Taylor series expansion, we have

$$H(s_1) = H(0) + H'(0)s_1 + \frac{1}{2!}H''(0){s_1}^2 + \frac{1}{3!}H'''(0){s_1}^3 + \frac{1}{4!}H^{(4)}(0){s_1}^4 + ... \quad (17)$$

$$K(s_2) = K(0) + K'(0)s_2 + \frac{1}{2!}K''(0){s_2}^2 + ... \quad (18)$$

$$B(s_3) = B(0) + B'(0)s_3 + ... \quad (19)$$

Finally, using (12)-(16) and the conditions

$H(0) = 1, H'(0) = 0, H''(0) = 0, H'''(0) = 6, \left|H^{(4)}(0)\right| < \infty$,

$K(0) = 1, K'(0) = 0, |K''(0)| < \infty$,

$B(0) = 1, B'(0) = 1.$

We get the error expression,

$$e_{n+1} = \gamma + \left(\frac{1}{24}H^{(4)}(0)c_2^5c_3 + K''(0)c_2c_3^3 - \frac{9}{2}c_2^5c_3 + 4c_2^3c_3^2 - c_2^2c_3c_4\right){e_n}^8 + O(e_n^9). \quad (20)$$

The theorem is proved.

Method 1: choosing

$H(s_1) = 1 + {s_1}^3 + (t{s_1}^a)\ , a \geq 3\ , a, t \in R$,

$K(s_2) = 1 + ({s_2}^\mu)\ , \mu > 1\ , \mu \in R$,

$B(s_3) = 1 + s_3$.

A new families of methods can be written as, (**HSM1**),

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{2f(x_n) - f(w_n)}{2f(x_n) - 5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{t{s_1}^a(s_3{s_2}^\mu + {s_2}^\mu + s_3 + 1) + {s_2}^\mu({s_1}^2s_3 + {s_1}^2 + 3) + ({s_1}^2 + 1)\cdot(s_3 + 1)\}\frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}, \quad (21)$$

Method 2: choosing

$H(s_1) = 1 + {s_1}^3$,

$K(s_2) = 1 + s_2\, sin(s_2)$,

$B(s_3) = 1 + s_3e^{s_3}$.

A new method can be obtained as, (**HSM2**),

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{2f(x_n) - f(w_n)}{2f(x_n) - 5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{{s_1}^3(s_2s_3e^{s_3}\, sin(s_2) + s_2\, sin(s_2) + s_3e^{s_3} + 1) + (s_2\, sin(s_2) + 1)(s_3e^{s_3} + 1)\}\frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}, \quad (22)$$

Method 3: choosing

$H(s_1) = 1 + {s_1}^3e^{s_1}$,

$K(s_2) = 1 - sin({s_2}^\mu),\ \mu > 1\ , \mu \in R$,

$B(s_3) = e^{s_3}$.

A new method can be written as, (**HSM3**),

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{2f(x_n) - f(w_n)}{2f(x_n) - 5f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{({s_1}^3e^{s_1+s_3} + e^{s_3})(1 - sin({s_2}^\mu))\}\frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}, \quad (23)$$

3. Numerical Results

In this section, we present some results of the numerical simulations are performed to compare the efficiencies of the present methods (HSM1 – HSM3) with the other well-known optimal eighth-order methods. We compared with the Newton method (NM), (1), the fourth–order method of King (KM), (2), and some optimal eighth order methods, method (SHM) proposed by Sharma in [12], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)},$

$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)}\right)^2\right]\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]},$ (24)

method (BWRM) proposed by Bi–Wu–Ren in [3], given by

$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$

$z_n = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\frac{f(y_n)}{f'(x_n)},$

$x_{n+1} = z_n - \frac{f(x_n)+(2+\theta)f(z_n)}{f(x_n)+\theta f(z_n)}\frac{f(z_n)}{f[z_n,y_n]+(z_n-y_n)f[z_n,x_n,x_n]},$ (25)

where θ ∈ R, method (BM8) proposed by Wang and Liu in [7], given by

$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$

$z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)},$

$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]}.$ (26)

All computations were done using MATLAB (R2017a) using 1000 digits, floating point (digits:= 1000). We have used as stopping criteria that $|x_n - \gamma| \leq 10^{-200}$and $|f(x_n)| \leq 10^{-200}$.

Displayed in Table 1, the test functions and their simple root. And in Table 2, the number of iterations denoted by (IT), and computes the value of $|f(x_n)|$ and $|x_n - \gamma|$. Moreover, computational order of convergence (COC) approximated as [14] also displayed in the Table 2,

$$\rho = \frac{ln|(x_{n+1}-\gamma)/(x_n-\gamma)|}{ln|(x_n-\gamma)/(x_{n-1}-\gamma)|}.$$

4. Conclusion

In this work, we presented a new class of eighth–order methods based on a special case of King's method. The methods has been developed by replacing $f'(z)$ using divided difference and equivalent construction of weighted functions to reduce the numbers of functions evaluation to four, to be optimal. Other methods using numerical examples are compared with the new class.

Table 1 Test functions and their simple root.

Functions	**Roots**
$f_1(x) = x^3 + 4x^2 - 15,$	$\gamma = 1.63198080556606$
$f_2(x) = \cos x - x,$	$\gamma = 0.739085133215161$
$f_3(x) = \log(x^4 + x + 1) + xe^x,$	$\gamma = 0.0$
$f_4(x) = \sin x - \frac{x}{3},$	$\gamma = 2.27886266007583$
$f_5(x) = \log(x) + \sqrt{x} - 5,$	$\gamma = 8.30943269423157$
$f_6(x) = x^3 + \log(1 + x),$	$\gamma = 0.0$
$f_7(x) = x^5 + x^4 + 4x^2 - 15,$	$\gamma = 1.3474280989683$

Remark: Choosing $a = 6, \mu = 6, t = 3$ for (HSM1), and $\mu = 2$ for (HSM3), in the examples.

Table 2 Numerical effects for the results of different functions.

Method	**IT**	$\boldsymbol{|f(x_n)|}$	$\boldsymbol{|x_n - \gamma|}$	**COC**
$f_1(x) = x^3 + 4x^2 - 15, x_0 = 2$				
NM	8	3.71811e-218	1.76667e-219	2
KM,β = 0	4	1.0251e-228	4.87079e-230	4

SHM	3	9.60798e-427	4.56524e-428	8
BRWM	3	7.38713e-467	3.51e-468	8
BM8	3	3.51621e-455	1.67073e-456	8
HSM1	3	6.83538e-473	3.24784e-474	8
HSM2	3	5.76404e-473	2.73879e-474	8
HSM3	3	1.90868e-479	9.0691e-481	8
Method	IT	$\|f(x_n)\|$	$\|x_n - \gamma\|$	COC
$f_2(x) = cos\, x - x, x_0 = 0.6$				
NM	8	3.00558e-379	1.79587e-379	2
KM,$\beta = 0$	4	1.10351e-349	6.5936e-350	4
SHM	3	9.49798e-674	5.67514e-674	8
BRWM	3	8.27062e-721	4.94178e-721	8
BM8	3	3.08586e-748	1.84383e-748	8
HSM1	3	8.6075e-687	5.14307e-687	8
HSM2	3	2.29797e-691	1.37306e-691	8
HSM3	3	1.07377e-687	6.41589e-688	8
$f_3(x) = log(x^4 + x + 1) + xe^x, x_0 = 0.25$				
NM	8	6.57994e-276	3.28997e-276	2
KM,$\beta = 0$	4	2.03822e-227	1.01911e-227	4
SHM	3	9.33255e-441	4.66628e-441	8
BRWM	3	1.53876e-428	7.69378e-429	8
BM8	3	4.79076e-442	2.39538e-442	8
HSM1	3	1.86404e-442	9.3202e-443	8
HSM2	3	1.03285e-387	5.16426e-388	8
HSM3	3	9.68341e-402	4.8417e-402	8
$f_4(x) = sin\, x - \frac{x}{3}, x_0 = 2$				
NM	7	3.1902e-249	3.24306e-249	2
KM,$\beta = 0$	4	7.18242e-210	7.30144e-210	4

SHM	3	9.83341e-386	9.99636e-386	8
BRWM	3	1.95602e-454	1.98843e-454	8
BM8	3	2.10587e-435	2.14077e-435	8
HSM1	3	9.28246e-430	9.43628e-430	8
HSM2	3	4.10166e-429	4.16963e-429	8
HSM3	3	8.30025e-440	8.43779e-440	8
$f_5(x) = log(x) + \sqrt{x} - 5, x_0 = 11.9$				
NM	8	1.14756e-218	3.90593e-218	2
KM,$\beta = 0$	4	1.87626e-237	6.38619e-237	4
SHM	3	3.92439e-492	1.33574e-491	8
BRWM	3	1.27296e-402	4.33275e-402	8
BM8	3	4.72438e-434	1.60803e-433	8
HSM1	3	9.93642e-421	3.38205e-420	8
HSM2	3	5.8735e-437	1.99916e-436	8
HSM3	3	8.99672e-403	3.0622e-402	8
$f_6(x) = x^3 + log(1 + x), x_0 = 0.25$				
NM	8	6.0375e-307	6.0375e-307	2
KM,$\beta = 0$	4	4.6847e-209	4.6847e-209	4
SHM	3	1.66015e-423	1.66015e-423	8
BRWM	3	5.55419e-411	5.55419e-411	8
BM8	3	1.29369e-444	1.29369e-444	8
HSM1	3	2.10179e-421	2.10179e-421	8
HSM2	3	9.59032e-340	9.59032e-340	8
HSM3	3	2.36056e-341	2.36056e-341	8
$f_7(x) = x^5 + x^4 + 4x^2 - 15, x_0 = 1.6$				
NM	9	4.61266e-319	1.24511e-320	2
KM,$\beta = 0$	5	1.57085e-741	4.24027e-743	4
SHM	3	1.90069e-340	5.13061e-342	8

BRWM	3	2.83009e-302	7.63936e-304	8
BM8	3	5.68394e-350	1.53429e-351	8
HSM1	3	1.12569e-336	3.03862e-338	8
HSM2	3	2.33297e-340	6.29747e-342	8
HSM3	3	2.33637e-341	6.30664e-343	8

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